LINK HOMOTOPY IN $S^n \times \mathbb{R}^{m-n}$ AND HIGHER ORDER μ–INVARIANTS

Ulrich Koschorke *

Universität Siegen, Emmy-Noether-Campus, D-57068 Siegen

Abstract. Given a suitable link map f into a manifold M, we constructed, in [10], link homotopy invariants $\kappa(f)$ and $\mu(f)$. In the present paper we study the case $M = S^n \times \mathbb{R}^{m-n}$ in detail. Here $\mu(f)$ turns out to be the starting term of a whole sequence $\mu^{(s)}(f)$, $s = 0, 1, \dots,$ of higher μ-invariants which together capture all the information contained in $\kappa(f)$. We discuss the geometric significance of these new invariants. In several instances we obtain complete classification results. A central ingredient of our approach is the homotopy theory of wedges of spheres.

1. Introduction

Given dimensions $p_1, \dots, p_r \geq 1$ and $m > n \geq 1$, we want to classify *link maps*

$$f = f_1 \amalg \cdots \amalg f_r : S^{p_1} \amalg \cdots \amalg S^{p_r} \longrightarrow M := S^n \times \mathbb{R}^{m-n} \tag{1.1}$$

(i.e. the spheres S^{p_j} have pairwise disjoint images under the continuous maps f_j , $1 \leq j \leq r$) up to *link homotopy* (i.e. continuous deformations through such link maps). Our approach centres around the homotopy class (in the standard sense)

$$\kappa_M(f) := [\widehat{f}] \in [S^{p_1} \times \cdots \times S^{p_r}, \widetilde{C}_r(M)] \tag{1.2}$$

where the product map $\widehat{f} := f_1 \times \cdots \times f_r$ takes values in the configuration space of ordered r-tuples of pairwise distinct points in M.

Unfortunately, this very natural link homotopy invariant lies in a rather unwieldy homotopy set. However, if f is κ_M*-Brunnian* (i.e. if $\widehat{f}$ is nulhomotopic when restricted to the complement of a point in $S^{p_1} \times \cdots \times S^{p_r}$), one can simplify (a base point preserving version of) $\kappa_M(f)$ considerably and then extract the "numerical" link homotopy invariant

$$\mu_M(f) \in \bigoplus^{(r-2)!} \pi^S_{p_1+\cdots+p_r-(r-1)(m-2)-1} \tag{1.3}$$

Key words and phrases. link map, link homotopy invariant, Hopf homomorphism, linking coefficient, stable suspension, configuration space.

* Supported within the German-Brazilian Cooperation by IB-BMBF and CNPq.

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

which generalizes e.g. Milnor's μ-invariants of classical links and, in particular, the classical linking number when $r = 2$. As the example of the higher dimensional Borromean link illustrates, $\mu_M(f)$ is in general weaker than $\kappa_M(f)$ (cf. [10], 5.9).

In the present paper we measure this loss of information. In Section 3 we show that $\mu_M(f)$ is only the starting element of a whole sequence $\{\mu_M^{(s)}(f)\}_{s\geq 0}$ of higher order μ-invariants which together characterize $\kappa_M(f)$ (cf. theorem 3.5). Furthermore we investigate the geometric meaning of the remaining higher invariants: they turn out to be the standard $\mu_{\mathbb{R}^m}$-invariants of the augmented link maps in $\mathbb{R}^m$ which consist of f, included into $\mathbb{R}^m$, together with a finite number of meridians $\{z_j\} \times S^{m-n-1}$ around $M \cong S^n \times \overset{\circ}{B}{}^{m-n}$ (cf. theorems 3.8 and 4.3).

The requirement that f is κ_M-Brunnian allows us to concentrate on linking phenomena of highest order, but it is quite restrictive. In particular we can define $\mu^{(s)}(f)$ only if all μ-invariants of all proper sub-link maps of f are defined and trivial. However, if $p_1, \ldots, p_r \leq m-3$ this restriction can be avoided: in section 4 we define a sum operation (conceivably without additive inverses) and use it to extend our invariants to *all* link maps whether they are κ_M-Brunnian or not. In particular, this allows us to introduce the *total higher μ-invariant* $\boldsymbol{\mu}(f)$ which consists of the homotopy classes $[f_j] \in \pi_{p_j}(S^n)$ of the component maps as well as of all (higher) μ_M-invariants of all sub-link maps $f_{j_1} \amalg \cdots \amalg f_{j_s}$, $2 \leq s \leq r$, of f.

Example: $p_1 = \cdots = p_r = 3$, $m = 6 > n \geq 1$. *Given* $r \geq 1$, *let* $BLM_{3,\ldots,3}(M_n)$ *denote the semigroup of all* (*base point preserving if* $n = 1$) *link homotopy classes of link maps*

$$f = \coprod_{j=1}^{r} f_j \quad : \quad \coprod^{r} S^3 \longrightarrow M_n := S^n \times \mathbb{R}^{6-n} .$$

Then for $n = 1$ *the total higher* μ-*invariant* $\boldsymbol{\mu}$ *is injective on*

$$BLM_{3,\ldots,3}(M_n) \cong \left(\bigoplus^{\infty} \mathbb{Z}_2\right)^{\binom{r}{2}} \oplus \left(\bigoplus^{\infty} \mathbb{Z}\right)^{\binom{r}{3}} ;$$

for $n > 1$, $\boldsymbol{\mu}$ *establishes the isomorphism*

$$BLM_{3,\ldots,3}(M_n) \cong \mathbb{Z}_2^{\binom{r}{2}} \oplus \mathbb{Z}^{\binom{r}{3}} \oplus \mathbb{Z}^a$$

where $a = \binom{r+1}{2}$, r *and* 0, *resp., for* $n = 2$, 3 *and* ≥ 4, *resp.*

Moreover, the obvious inclusions $M_n \subset M_{n+1}$ *induce homomorphisms which annihilate* $\mathbb{Z}^a$ *and correspond to summation or to the identity on the remaining direct summands.*

This will be proved in section 5 where we compare quite generally our invariants to linking coefficients of embedded links in codimensions greater than 2. It turns out that the transition from link isotopy to link homotopy translates into applying Freudenthal suspensions to crucial components of the linking coefficients. (In view of the close relation between the exact link homotopy sequence and James' EHP-sequence (cf. [8], theorem 3.1) this comes as no surprise).

Such investigations open the way to further injectivity, surjectivity, and nontriviality results for our invariants and also to criteria deciding which aspects of the linking coefficient depend only on the link homotopy class of an embedded link.

Example: $n = 1,\ r = 2,\ p_1, p_2 \leq m-3$. Assume that $\pi_{p_2}(S^{m-p_1-1})$ is stable (i.e. the stable suspension is bijective). Then the total higher μ-invariant $\boldsymbol{\mu}$ establishes an isomorphism from the semigroup consisting of all base point preserving link homotopy classes of those link maps $f_1 \amalg f_2 \ :\ S^{p_1} \amalg S^{p_2} \longrightarrow S^1 \times \mathbb{R}^{m-1}$ where f_1 is a smooth embedding, onto $\pi_{p_1}(S^1) \oplus \pi_{p_2}(S^1) \oplus \bigoplus^{\infty} \pi^S_{p_1+p_2-m+1}$.

If we add "parallel longitudes" to an embedded link f our techniques allow us also to capture certain components of the linking coefficient of f which are not invariant under link homotopy. Details will be given elsewhere. Let us just note here that this approach leads sometimes to a full *isotopy* classification. E. g. if $2 \leq p \leq \frac{2}{3}(m-2)$ it follows from [3], Corollary 1.3 (compare also [5], § 4) that any smooth embedding $f_1 : S^p \hookrightarrow S^1 \times \mathbb{R}^{m-1}$ which is nullisotopic when included into $\mathbb{R}^m$, is already determined up to isotopy by $\boldsymbol{\mu}(f_1 \amalg f_1^+)$ where f_1^+ is such a parallel longitude.

For our analysis of κ-invariants and of linking coefficients we need a good geometric understanding of the homotopy groups of wedges of spheres. In section 2 we study these not just via the "incoming" Hilton isomorphism but mainly via "outgoing" Hopf homomorphisms which enjoy many compatibility properties and are more suited to detect relevant geometric aspects of our invariants.

Notations and Conventions. From now on we always assume $r \geq 2,\ m \geq 3,\ p_1, \ldots, p_r \geq 1$. Σ_s and $E^{(\infty)}$ denote the permutation group in s elements and (stable) suspension, resp. All spheres and their wedges are equipped with base points. Homotopy classes in such wedges are identified, via Pontryagin-Thom, with bordisms classes of framed links.

2. Pinching and Hopf homomorphisms

Throughout this section let $n \geq 1$ and $r,\ q_1, \ldots\ q_{r-1} \geq 2$ be natural numbers; put

$$|q| \ :=\ q_1 \ + \cdots +\ q_{r-1}\ . \tag{2.1}$$

We will extend the geometric Hopf homomorphism discussed e.g. in § 3 of [9] in order to study the homotopy groups of the wedge

$$W \ :=\ S^n \vee \bigvee_{j=1}^{r-1} S_j^{q_j} \tag{2.2}$$

of spheres of the indicated dimensions. More specifically, in view of later applications we will be interested in certain subgroups such as the *reduced* groups

$$\pi'_k(W) \ :=\ \bigcap_{i=1}^{r-1} \ker\left(\pi_k(W) \ \longrightarrow\ \pi_k(S^n \vee \bigvee_{\substack{j=1\\ j\neq i}}^{r-1} S_j^{q_j})\right), \tag{2.3}$$

$k \in \mathbb{Z}$, defined by the obvious collapsing maps.

Definition 2.4. Given an integer $s \geq 0$, a permutation γ of the set $\{1, \ldots, s, \ldots, r+s-2\}$ is called *s-monotone* if $\gamma^{-1}(1) < \cdots < \gamma^{-1}(s)$.

Let $\sum_{r,s} \subset \sum_{r+s-2}$ denote the subset of s-monotone permutations in the full permutation group of $r+s-2$ elements.

Since the s-monotone permutations form a system of representatives of the cosets in $\sum_s \setminus \sum_{r+s-2}$ we have

$$u_{r,s} \ := \ |\sum_{r,s}| \ = \ (r+s-2)!/s! \tag{2.5}$$

Given $s \geq 0$, consider also the s-fold pinch map

$$\text{pinch}_s \ : \ (S^n, *) \ \longrightarrow \ \bigvee_{j=1}^{s} (S^n_j, *)$$

which has degree 1 when collapsed to any wedge summand S^n_j (and, if $n=1$, traverses the circles S^n_j in the order given by the subindex j); these requirements determine pinch_s uniquely up to homotopy. For every permutation $\gamma \in \sum_{r+s-2}$ we define the homomorphism

$$H_{s,\gamma} \ : \ \pi_k(W) \ \longrightarrow \ \pi^S_{k-sn-|q|+r+s-2} \quad , \tag{2.6}$$

$k \in \mathbb{Z}$, by $H_{s,\gamma} = h_\gamma \circ (\text{pinch}_s \vee \text{id})_*$ where h_γ is the Hopf invariant described in § 3 of [9]. Summing over all *s-monotone* permutations γ we obtain

$$H_s \ := \bigoplus_{\gamma \in \sum_{r,s}} H_{s,\gamma} \quad : \quad \pi_k(W) \longrightarrow (\pi^S_{k-sn-|q|+r+s-2})^{u_{r,s}} \quad . \tag{2.7}$$

We define the *total Hopf homomorphism* H by the direct product

$$H \ := \prod_{s \geq 0} H_s \quad : \quad \pi_k(W) \longrightarrow \prod_{s \geq 0} (\pi^S_{k-sn-|q|+r+s-2})^{u_{r,s}} \quad . \tag{2.8}$$

Now consider first *the case when* $n \geq 2$ (then the above product is actually a finite direct sum).

Given $s \geq 0$, the homotopy class of pinch_s remains unchanged if we permute the wedge summands in $\vee^s S^n_j$; thus using only s-monotone permutations γ in 2.7 above helps us to avoid unnecessary redundancies.

Moreover, we choose a Hilton decomposition of $\pi_*(W)$ (cf. [6]) and we define

$$\pi''_k(W) \ := \ \bigoplus_{t \geq 0} \pi''_k(W,t) \quad \subset \pi'_k(W), \quad k \in \mathbb{Z} \ , \tag{2.9}$$

where for any integer $t \geq 0$ $\pi''_k(W,t)$ denotes the direct summand corresponding to all basic Whitehead products which involve the inclusion $\iota_0 : S^n \subset W$ precisely t times and each inclusion $\iota_j : S^{q_j}_j \subset W$ precisely once, $j = 1, \ldots, r-1$.

Note that any such iterated Whitehead product w, basic or not, is a $\mathbb{Z}$–linear combination of Whitehead products of the form

$$\iota_\delta \ := \ [\iota_{\delta(1)}, [\iota_{\delta(2)}, [\ldots, [\iota_{\delta(r+t-2)}, \iota_{r-1}]]]] \tag{2.10}$$

where the same factors occur the same number of times, bracketed in the indicated special fashion and arranged in the order prescribed by a map $\delta \ : \ \{1, ..., r+t-2\} \to \{0; 1, \ldots, r-2\}$. This follows inductively from the Jacobi identity. Indeed, if w contains a subproduct of the form $[[a,b], w_1]$ where $\pm w_1$ is the longest subproduct involving ι_{r-1} and bracketed as in 2.10, then we may replace $[[a,b], w_1]$ by $\pm[a, [b, w_1] \pm [b, [a, w_1]]$. We repeat such substitutions until ι_{r-1} occurs only in *longer* subproducts $[\iota_\ell, w_1]$ which are bracketed as in 2.10.

Now we can evaluate our Hopf homomorphisms.

Proposition 2.11. *Assume* $n \geq 2$*. Given any integers* $s, t \geq 0$*, let* $\gamma \in \sum_{r+s-2}$ *be a permutation and let* $\delta : \{1, \ldots, r+t-2\} \to \{0; 1, \ldots, r-2\}$ *be a map such that* $\delta^{-1}(\{j\})$ *consists of precisely one element for* $j = 1, \ldots, r-2$*.*

If $s = t$ *and* δ *is the "contraction" of* γ *(i.e.* $\delta(i) = \max(\gamma(i) - t, 0)$ *for* $i = 1, \ldots, r+t-2$*), then for all* $k \in \mathbb{Z}$ *the composite map*

$$\pi_k(S^{tn+|q|-r-t+2}) \xrightarrow{\iota_{\delta *}} \pi_k(W) \xrightarrow{H_{s,\gamma}} \pi^S_{k-sn-|q|+r+s-2}$$

(cf. 2.6 *and* 2.10*) equals the stable suspension* E^∞ *up to a* $\pm$*–sign.*

In all other cases $H_{s,\gamma} \circ \iota_{\delta *} \equiv 0$*.*

Moreover, for $s \geq 0$ H_s *vanishes on all those Hilton summands of* $\pi_*(W)$ *which are not contained in* $\pi''_*(W, s)$*.*

Proof. If $s \leq t$, the first two claims follow from arguments (based e.g. on the fibrewise intersection approach to Hopf invariants) similar to those in the proof of theorem 3.1 in [9]. Pinching (or, equivalently, replacing certain submanifolds by several "parallel" copies) adds no difficulties since all codimensions are strictly larger than 1.

If $s > t$, then the intersection approach to $H_{s,\gamma}(\iota_\delta)$ requires two steps. At first we carry out the iterated intersection procedure described in [9], pp. 306–307, until we are confronted with the overcrossing locus

$$g|N_{\gamma(1)} \ > \cdots > \ g|N_{\gamma(s-t)} \quad > \bar{g}|\mathcal{I}$$

as in [9], (12). Here $\mathcal{I}$, if non empty, consists of a single point and hence is not nulbordant. But $N_{\gamma(1)}$ has a standard nulbordism B; thus we may replace overcrossings again by intersections, but in the reverse order, i.e. starting with $B \cap N_{\gamma(2)}$ etc. All necessary intermediate nulbordisms exist (since no $N_{\gamma(i)}$ corresponds to ι_{r-1}) and they have strictly positive codimensions so that they miss $\mathcal{I}$ generically. As in [9], top of p. 309, this argument can be applied fibre by fibre to show that $H_{s,\gamma} \circ \iota_{\delta_*} \equiv 0$.

Finally, if a Whitehead product ι_δ as in 2.10 involves a factor ι_j for some j, $1 \leq j \leq r-2$, more than once then $H_{s,\gamma} \circ \iota_{\delta *} \equiv 0$ since in the iterated (standard) intersection process $H_{s,\gamma}$ can register only the "innermost" factor ι_j and treats all outer ones as if they were zero. With a little extra care such an argument can also be adapted to $j = r-1$. In view of the discussion following 2.10 we conclude that $H_{s,\gamma} \circ w_* \equiv 0$ for every basic Whitehead product $w \in \pi_*(W)$ which involves some factor ι_j, $1 \leq j \leq r-1$, more than once (or not at all) or which involves ι_0 t times, $t \neq s$. ■

Theorem 2.12. *Assume* $n \geq 2$*. Given any integers* k *and* $s \geq 0$*, the (restricted) Hopf homomorphism* $H_s| = H_s|\pi''_k(W; s)$ *(cf.* 2.7 *and* 2.9*) fits into the commuting diagram* *where* $\sum w_{\ell *}$ *is the Hilton isomorphism,* E^∞ *denotes stable suspension and* D_{s*} *is defined by the operation of an invertible matrix* $D_s \in GL(u_{r,s};\mathbb{Z})$.

In particular, if $k \leq 2(|q|-r+1+s(n-1))$, *then* $H_s|$ *is bijective.*

If even $k \leq 2(|q|-r+1)$, *then* H (*cf.* 2.8) *restricts to the isomorphism* (*cf.* 2.9)

$$H| \ : \ \pi''_k(W) \xrightarrow{\ \cong\ } \bigoplus_{s\geq 0} \left(\pi^S_{k-|q|+r-2-s(n-1)}\right)^{u_{r,s}} .$$

Proof. In view of [6], p. 155, the number of basic Whitehead products w_ℓ generating $\pi''_k(W;s)$ turns out to coincide with $u_{r,s}$ (compare 2.5). Up to transposition and $\pm$ signs, the $u_{r,s}\times u_{r,s}$–matrix D_s consists of the integer coefficients encountered when we express these w_ℓ in terms of the elements ι_δ as in 2.10 (for $t=s$) or, equivalently, of their $\pi''_*(W,s)$–components. But these, in turn, are $\mathbb{Z}$–linear combinations of the basic Whitehead products w_ℓ. Thus D_s is invertible.

As in 2.11 we evaluate the Hopf invariants H_s fibrewise via the intersection approach (compare [9], p. 308–309); this yields our commutativity claim. ∎

It remains to discuss *the case when* $n=1$.

Let $\widetilde{W}$ denote the universal covering space of W. We may think of it as a real line $\mathbb{R}$, with a wedge $\widetilde{W}_g = \bigvee_{j=1}^{r-1} S^{q_j}_{j,g}$ attached at every integer $g\in\mathbb{Z}\subset\mathbb{R}$. We write

$$c \ : \ \widetilde{W} \longrightarrow \mathbb{R} \tag{2.13}$$

for the obvious "level" map. It is compatible with the "shifts" (deck transformations) by elements in $\mathbb{Z}=\pi_1(W)=\pi_1(S^1)$.

On the other hand, there is a canonical isomorphism $\pi_*(\widetilde{W})\cong\pi_*(\vee_g\widetilde{W}_g)$ and we may choose a Hilton decomposition based on Whitehead products which involve the inclusions $\iota_{j,g} : S^{q_j}_{j,g}\subset\vee_g\widetilde{W}_g$ (where $1\leq j\leq r-1$ and $g\in\mathbb{Z}$).

This applies also to our original wedge W via the isomorphism

$$p_* \ : \ \pi'_*(\widetilde{W}) \xrightarrow{\ \cong\ } \pi'_*(W) \tag{2.14}$$

of *reduced* homotopy groups (compare 2.3 and also [10], 2.6 and 2.10) where p denotes the covering projection. Then $\pi'_*(W)$ is the direct sum of all Hilton summands corresponding to basic Whitehead products which for each $j=1,\ldots,r-1$ involve at least one factor of the form $\iota_{j,g}, g\in\mathbb{Z}$. Define the subgroup

$$\pi''_*(W) \ \subset \ \pi'_*(W) \tag{2.15}$$

to be the direct sum of all such summands which involve *precisely* one such factor for each j.

Now let us evaluate our Hopf homomorphisms on an arbitrary class $[v]\in\pi'_k(W)$. We adopt the intersection approach outlined in [9], pp. 306–307.

FIGURE 2.16. The arrangement of the manifolds $N_{j,g}$ and $M_g(i)$ (Here link components and other subsets of $\mathbb{R}^k$ are listed, together with their images in $\mathbb{R}$ under $c\,\widetilde{v}$)

Let $\widetilde{v} : S^k \to \widetilde{W}$ be the lifting of a generic representative v. Then for $j = 1, \dots, r-1$ the link component

$$N_j \;=\; v^{-1}(\{z_j\}) \;\subset\; \mathbb{R}^k \;=\; S^k - \{*\}$$

corresponding to $S_j^{q_j}$ is the disjoint union of the manifolds

$$N_{j,g} \;=\; \widetilde{v}^{-1}(\{g\widetilde{z}_j\}), \quad g \in \mathbb{Z}\ ; \tag{2.17}$$

here $z_j \in S_j^{q_j} - \{*\}$ and $g\widetilde{z}_j \in S_{j,g}^{q_j} - \{*\}$ are regular values of v and $\widetilde{v}$. Similarly the link component $M = v^{-1}(\{z_0\})$, $z_0 = (-1, 0) \in S^1 - \{*\} \subset W$, is the disjoint union of the hypermanifolds $M_g = \widetilde{v}^{-1}(\{g - \frac{1}{2}\})$, $g \in \mathbb{Z}$, which decompose $\mathbb{R}^k$ into the strips $Q_g = (c\widetilde{v})^{-1}[g - \frac{1}{2}, g + \frac{1}{2}]$. Note that Q_0 is unbounded and that

$$N_{j,g} \;=\; Q_g \cap N_j \quad \text{for } j = 1, \dots, r-1 \text{ and all } g \in \mathbb{Z}\ .$$

For $s \geq 0$ pinching S^1 s times amounts to replacing M (and M_g) by s nearby "parallel" copies $M(i)$ (and $M_g(i)$), $i = 1, \dots, s$, (where e.g.

$$M_g(i) \;=\; (c\widetilde{v})^{-1}(\{g - \frac{1}{2} + i\varepsilon\})\ , \quad g \in \mathbb{Z}\ , \tag{2.18}$$

for some small fixed $\varepsilon > 0$, compare figure 2.16).

Given an s–monotone permutation $\gamma \in \sum_{r,s}$ (cf. 2.4), it will be convenient to calculate the Hopf invariant $H_{s,\gamma}[v]$ via iterated intersections as in [9], (12), but in the *reverse order* (this changes the value only by a fixed sign which is obvious from the overcrossing description of $H_{s,\gamma}[v]$, cf. p. 306 of [9]). Thus if $N(\gamma(1)), N(\gamma(2)), \dots, N(\gamma(r+s-2))$, N_{r-1} are the link components of $(\mathrm{pinch}_s \vee \mathrm{id})_*[v]$, listed in the order given by γ, intersect a framed (singular) nulbordism of $N(\gamma(1))$ in $\mathbb{R}^k$ with $N(\gamma(2))$ to obtain a closed framed singular manifold $N(\gamma(1), \gamma(2))$; similarly, intersect a nulbordism of $N(\gamma(1), \gamma(2))$ in $\mathbb{R}^k$

with $N(\gamma(3))$, etc., until finally a nulbordism of $N(\gamma(1),\dots,\gamma(r+s-2))$ is intersected with N_{r-1} to yield $\pm H_{s,\gamma}[v]$.

Since by assumption $[v]$ lies in the reduced homotopy group $\pi'_k(W)$, all intermediate intersections which occur in this process are indeed nulbordant. In fact, often we have even canonical nulbordisms. E.g. $M_g(i)$ bounds the compact manifold $B_g(i)$ defined to be $(c\widetilde{v})^{-1}[g-\frac{1}{2}+i\varepsilon,+\infty)$ if $g\geq 1$, or, with opposite sign, $(c\widetilde{v})^{-1}(-\infty,g-\frac{1}{2}+i\varepsilon]$ if $g\leq 0$. Thus $B(i)=\bigcup_{g\in\mathbb{Z}} B_g(i)$ is a nulbordism of $M(i)$ which, for every $g\in\mathbb{Z}$, covers most of the strip Q_g – and hence all of $N_{j,g}$, $j=1,\dots,r-1$, – precisely g times. Figure 2.16 may help to visualize such statements.

Similarly, $B_g(i)\cap B_{g'}(i')$ is a nulbordism of $\pm B_g(i)\cap M_{g'}(i')$ if $1<g+i\varepsilon<g'+i'\varepsilon$ or $1>g+i\varepsilon>g'+i'\varepsilon$ etc.

Example 2.19: r = 2. Here the generic base point preserving map

$$\widetilde{v}\ :\ S^k\ \longrightarrow\ \widetilde{W}\sim\bigvee_{g\in\mathbb{Z}} S^{q_1}_{1,g}$$

corresponds to the (finite) framed link

$$N_1\ =\coprod_{g\in\mathbb{Z}} N_{1,g}\ \subset\ \mathbb{R}^k=S^k-\{*\}$$

For each $g\in\mathbb{Z}$ the (stable) framed bordism class

$$[N_{1,g}]\ \in\ \Omega^{fr}_{k-q_1}\ \cong\ \pi^S_{k-q_1}$$

is given by the stable suspension of the composite of $\widetilde{v}$ with the map which collapses all wedge summands $S^{q_1}_{1,g'}$, $g'\neq g$, to a point.

For each $s\geq 0$ there is only one s–monotone permutation $\gamma\in\sum_{2,s}$. We can apply the previous discussion to the resulting Hopf homomorphism $H_s=H_{s,\gamma}$ and obtain

$$\pm H_s([v])\ =\sum_{g\in\mathbb{Z}}\binom{g+s-1}{s}[N_{1,g}]\ \in\pi^S_{k-q_1}\quad . \tag{2.20}$$

Indeed, $H_0([v])$ does not involve any of the hypermanifolds $N_g(i)$ and $\pm H_0([v])$ is just equal to $[N_1]=\sum[N_{1,g}]$. Next,

$$\pm H_1([v])\ =\ [B(1)\cap N_1]\ =\ \sum_{g'}[B_{g'}(1)\cap N_1]$$

counts each link component $N_{1,g}$ precisely g times since it is contained in just so many 0–codimensional bordisms $B_{g'}(1)$ (which have the opposite orientations for $g'\leq 0$ so that the signs are also correct). In the same spirit but more generally, let $b_{s,g}$ denote the integer which counts the relevant s–fold intersections $B_{g_1}(1)\cap\cdots\cap B_{g_s}(s)$ containing $N_{1,g}$, $s\geq 1$, $g\in\mathbb{Z}$. (By "relevant" we mean that the intersection can contribute to $H_s[v]$, i.e. $B_{g_i}(i)\supset M_{g_{i+1}}(i+1)$ for $1\leq i<s$). Comparing such intersections to those which contain $N_{1,g-1}$, we see that

$$b_{s,g}\ =\ b_{s,g-1}\ +\ b_{s-1,g} \tag{2.21}$$

(figure 2.16 may again be helpful here). The identity

$$b_{s,g} \;=\; \binom{g+s-1}{s} \tag{2.22}$$

and formula 2.20 follow now by induction over s (and, for fixed s, over $|g|$). ■

The next result shows that the sequence $\{[N_{1,g}]\}_{g\in\mathbb{Z}}$ is completely determined by the sequence $\{H_s([v])\}_{s\geq 0}$ of Hopf invariants.

Lemma 2.23. *For any abelian group A the homomorphism*

$$d \;=\; \prod_{s\geq 0} d_s \;:\; \bigoplus_{g\in\mathbb{Z}} A \;\longrightarrow\; \prod_{s\geq 0} A \;\;,$$

defined by $d_s((a_g)_{g\in\mathbb{Z}}) \;=\; \sum_{g\in\mathbb{Z}} \binom{g+s-1}{s} a_g$, *is injective.*

Proof. Given any integers $n\geq 0$ and n_0, consider the $(n+1)\times(n+1)$–matrix $M(n,n_0)$ with entries

$$b_{s,g} \;=\; \binom{g+s-1}{s}, \quad 0\leq s\leq n, \quad n_0-n\leq g\leq n_0 \;.$$

When we subtract the n^{th} column from the $(n+1)^{\text{st}}$ one, the $(n-1)^{\text{st}}$ column from the n^{th} column and so forth, the top row takes the form $(1,0,\dots,0)$ and – in view of 2.21 – the $n\times n$–submatrix in the lower right hand corner coincides with $M(n-1,n_0)$. Thus

$$\det M(n,n_0) \;=\; \det M(n-1,n_0) \;=\cdots=\; \det M(0,n_0) \;=1\;.$$

Since every element of $\ker d$ lies already in the kernel of the endomorphism $M(n,n_0)$ on $\bigoplus_{n_0-n\leq g\leq n_0} A$ for suitable n and n_0, our lemma follows. ■

Next let $r\geq 2$ be arbitrary. Given an $(r-1)$–tuple $(g)=(g_1,\dots,g_{r-1})\in\mathbb{Z}^{r-1}$ of integers and a permutation $\bar\gamma\in\sum_{r-2}$, let

$$h_{(g),\bar\gamma} \;:\; \pi'_k(W) \;\longrightarrow\; \pi^S_{k-|q|+r-2} \tag{2.24}$$

be the homomorphism which maps a homotopy class $[v]\in\pi'_k(W)$ or, equivalently, the corresponding framed link $\amalg\, N_{j,g}\subset\mathbb{R}^k$ as in 2.17, to the Hopf invariant (cf. [9], § 3)

$$h_{\bar\gamma}\;\left(N_{1,g_1}\;\amalg\;\dots\;\amalg\;N_{r-1,g_{r-1}}\right)$$

of the indicated sublink (on the homotopy level the selection of this sublink is induced by the obvious map which collapses $\widetilde{W}$ to $\bigvee_{j=1}^{r-1} S^{q_j}_{j,g_j}$, compare the discussion preceding 2.14).

Now, for $s\geq 0$, consider an s–monotone permutation $\gamma\in\sum_{r,s}$. Then the list of values of γ takes the form

$$(1,\dots,s_1;\bar\gamma(1){+}s;s_1{+}1,\dots,s_1{+}s_2;\bar\gamma(2){+}s;\dots;\bar\gamma(r{-}2){+}s;s_1{+}\cdots{+}s_{r-2}{+}1,\dots,s) \tag{2.25}$$

where $(s)=(s_1,\dots,s_{r-1})\in\mathbb{Z}_+^{r-1}$ is an $(r-1)$–tuple of nonnegative integers whose sum equals s, and $\bar\gamma\in\sum_{r-2}$; in words: s_i (strictly increasing) values lying in $\{1,\dots,s\}$, $i=1,\dots,r-1$, alternate with the values $\bar\gamma(i){+}s\in\{s{+}1,\dots,r{+}s{-}2\}$ which appear in the order given by the permutation $\bar\gamma$. We obtain a bijection between such pairs $((s),\bar\gamma)$ and the corresponding s–monotone permutations $\gamma((s),\bar\gamma)\;\in\;\sum_{r,s}$ defined by 2.25.

Proposition 2.26. *For* $\gamma = \gamma((s), \bar{\gamma})$ *as above and for every homotopy class* $[v] \in \pi'_k(W)$, $k \in \mathbb{Z}$, (*cf.* 2.3) *we have*

$$H_{s,\gamma}([v]) \;=\; \varepsilon \cdot \sum_{(g)=(g_1,\dots,g_{r-1})\in\mathbb{Z}^{r-1}} \; \prod_{j=1}^{r-1} \binom{\bar{g}_j - \bar{g}_{j-1} + s_j - 1}{s_j} \, h_{(g),\bar{\gamma}}\;([v])$$

where $\varepsilon = \pm 1$ *is a fixed sign depending only on* γ *and* k, *and* $\bar{g}_j := g_{\bar{\gamma}(j)}$ *for* $j = 1, \dots, r-2$ *(and* $\bar{g}_0 := 0$ *and* $\bar{g}_{r-1} := g_{r-1}$*).*

Proof. Adopting the intersection approach (see the discussion which follows 2.18) we iterate the procedure used in example 2.19 (compare 2.20). After the first step (i.e. after the s_1–fold intersection with "standard" 0–codimensional nulbordisms), each link component $N_{\bar{\gamma}(1),\bar{g}_1}$ (cf. 2.18), $\bar{g}_1 \in \mathbb{Z}$, is counted with the multiplicity $b_{s_1,\bar{g}_1}$ (cf. 2.22).

If $r > 2$ then each $N_{\bar{\gamma}(1),\bar{g}_1}$, in turn, allows a (generic, singular) nulbordism B in $\mathbb{R}^k$ (use 2.15 and the fact that $[v]$ is reduced). Since $c \circ \widetilde{v}$ maps ∂B to $\bar{g}_1$ (cf. figure 2.16), $B \cap M_g(i)$ (cf. 2.18), $g \in \mathbb{Z}$, bounds the intersection of B with $(c\widetilde{v})^{-1}[g - \frac{1}{2} + i\varepsilon, \infty)$ if $g > \bar{g}_1$, and with $(c\widetilde{v})^{-1}(-\infty, g - \frac{1}{2} + i\varepsilon]$ if $g \leq \bar{g}_1$. In the second step of our iteration we have to intersect B s_2–times in this fashion and then with $N_{\bar{\gamma}(2)}$. This results in $B \cap N_{\bar{\gamma}(2),\bar{g}_2}$, $\bar{g}_2 \in \mathbb{Z}$, being counted with the (additional) multiplicity $b_{s_2,\bar{g}_2-\bar{g}_1}$ (use the same arguments as in example 2.19, but with g replaced by $g - \bar{g}_1$).

Continuing this process and intersecting with N_{r-1} in the last step, we obtain the indicated linear combination of the Hopf invariants $h_{(g),\bar{\gamma}}[v]$. Each of these can be evaluated via the intersection approach since $[v]$ is reduced and hence all (intermediate intersections involving) sublinks with strictly less than $r-1$ components $N_{j,g}$ allow nulbordisms (compare 2.14). ■

Now we are ready to establish the analogue of theorem 2.12.

Theorem 2.27. *Assume* $n = 1$*. Then for all integers* k *the Hopf homomorphism* H (*cf.* 2.8)*, when restricted to* $\pi''_k(W)$ (*cf.* 2.15)*, fits into the commuting diagram*

where $\sum w_{\ell *}$ *is the Hilton isomorphism* (*cf.* 2.15)*,* E^∞ *denotes stable suspension and* D *is a group monomorphism. Moreover,* H *vanishes on all those Hilton summands of* $\pi_k(W)$ *which are not contained in* $\pi''_k(W)$*.*

In particular, if $k \leq 2(|q| - r + 1)$*, then* H *is injective on* $\pi''_k(W)$*.*

Proof. According to the last proposition there is a commuting diagram

where the range of H is indexed by all s–monotone permutations, $s \geq 0$, in the form $\gamma = \gamma((s), \bar{\gamma})$ (compare 2.25) and D' is defined by the $\mathbb{Z}$–linear combinations in 2.26.

First we want to prove that D' – or, equivalently, $D'_{\bar{\gamma}}$ for each $\bar{\gamma} \in \sum_{r-2}$ – is a monomorphism. After reparametrizing suitably we must show this only for the map D'_{r-1} defined by

$$D'_{r-1}((a_{(g)})_{(g)\in\mathbb{Z}^{r-1}}) = \left(\sum_{(g)} \prod_{j=1}^{r-1} \binom{g_j + s_j - 1}{s_j} a_{(g)} \right)_{(s)\in\mathbb{Z}_+^{r-1}} .$$

Thus suppose that

$$a \;=\; (a_{(g)})_{(g)=(g_1,\ldots,g_{r-1})\in\mathbb{Z}^{r-1}}$$

lies in the kernel of D'_{r-1}. Then for every $(s) = (s_1, \ldots, s_{r-1}) \in \mathbb{Z}_+^{r-1}$ the expression

$$\sum_{g_{r-1}\in\mathbb{Z}} \binom{g_{r-1} + s_{r-1} - 1}{s_{r-1}} \left(\sum_{(g_1,\ldots,g_{r-2})\in\mathbb{Z}^{r-2}} \prod_{j=1}^{r-2} \binom{g_j + s_j - 1}{s_j} a_{(g_1,\ldots,g_{r-2},g_{r-1})} \right)$$

vanishes, and so does the sum to the right hand side, for every fixed $g_{r-1} \in \mathbb{Z}$, by lemma 2.23. Note that such sums, indexed by $(s_1, \ldots, s_{r-2})$, constitute the value of $a_{(-,\ldots,-,g_{r-1})}$ under D'_{r-2}. Thus we can iterate our argument until the injectivity of $D'_1 = d$ (cf. 2.23) implies that $a = 0$.

Next we have to compare, for fixed $(g) \in \mathbb{Z}^{r-1}$, the values of $h_{(g),\bar{\gamma}}$, $\bar{\gamma} \in \sum_{r-2}$, to the stable suspensions of the Hilton components which correspond to the basic Whitehead products in $\iota_{1,g_1}, \ldots, \iota_{r-1,g_{r-1}}$. When we deal with $\pi''_k(W)$, each of these factors is involved precisely once, and the comparison can be expressed via an invertible matrix D_0 as in theorem 2.12. Composing $\bigoplus_{(g)} D_{0*}$ with D' we obtain the desired monomorphism D in 2.27.

Finally, consider a basic Whitehead product which, for some $1 \leq j \leq r-1$, involves more than one factor of the form $\iota_{j,g}$, $g \in \mathbb{Z}$ (or none at all). As in the proof of proposition 2.11 we conclude that $H \circ w_* \equiv 0$. Clearly, also $H|\pi_1(W) \equiv 0$. ■

In view of theorems 2.12 and 2.27 it is of interest to compare the subgroups $\pi'_k(W)$ and $\pi''_k(W)$ of $\pi_k(W)$, $k \in \mathbb{Z}$ (cf. 2.3, 2.9, and 2.15).

Lemma 2.28. *Assume* $n \geq 1$. *If*

$$k \ \leq \ |q| \ + q_j - r \quad \textit{for } j = 1, \ldots, r-1 \ ,$$

then $\pi'_k(W) = \pi''_k(W)$.

Indeed, $\pi''_k(W)$ (or $\pi'_k(W)$, resp.) is the direct sum of the Hilton summands corresponding to those basic Whitehead products which for every $j = 1, \ldots, r-1$ involve the inclusion $\iota_j \ : \ S^{q_j} \subset W$ (if $n \geq 2$) or an inclusion of the form $\iota_{j,g} \ : \ S^{q_j} \subset \widetilde{W}$ (if $n = 1$) precisely once (or at least once, resp.). In the indicated dimension range the additional Hilton summands in $\pi'_k(W)$ are all trivial.

3. Higher μ-invariants and their geometry

From now on let M denote the manifold $S^n \times \mathbb{R}^{m-n}$ (where $1 \leq n < m \geq 3$) unless mentioned otherwise. Pick a suitable base point $y^0 = (y_1^0, \ldots, y_r^0)$ of the configuration space

$$\widetilde{C}_r(M) \ = \{(y_1, \ldots, y_r) \in M^r \mid y_i \neq y_j \ \text{ for } 1 \leq i \neq j \leq r\} \tag{3.1}$$

Suitable embeddings $S^n = S^n \times \{x_0\} \subset M$ and $(\bigvee^{r-1} B_j^m, *) \subset (M, y_r^0)$ (such that the base point y_j^0 lies in the interior of the ball B_j^m, $j = 1, \ldots r-1$) yield a homotopy equivalence

$$S^n \vee \bigvee_{j=1}^{r-1} S_j^{m-1} \ \hookrightarrow \ M - \{y_1^0, \ldots, y_{r-1}^0\}$$

which is canonical up to homotopy (if in the case $n = m-1$ we also require $x_0 \in \mathbb{R}$ to be so small that $y_1^0, \ldots, y_{r-1}^0 \in S^{m-1} \times (x_0, \infty)$). Together with the "fibre inclusion"

$$M - \{y_1^0, \ldots, y_{r-1}^0\} \ \longrightarrow \ \widetilde{C}_r(M), \quad y \longrightarrow (y_1^0, \ldots, y_{r-1}^0, y) \ ,$$

and an obvious quotient map this induces the injective composite map

(3.2)

(compare [10], 2.5) where

$$|p| \ := \ p_1 + \cdots + p_r \quad . \tag{3.3}$$

Now we are ready to define higher μ-invariants. First we work in a setting where the base points $* \in S^{p_j}$ are preserved by link maps and link homotopies. Later we will also comment on the question as to when the resulting invariants remain unchanged under base point free link homotopies.

Thus let

$$f \ = \ f_1 \amalg \cdots \amalg f_r \ : \ S^{p_1} \amalg \cdots \amalg S^{p_r} \ \longrightarrow \ S^n \times \mathbb{R}^{m-n}$$

be a κ_M-Brunnian link map which, in addition, preserves base points, i.e. $f_j(*) = y_j^0$ for $j = 1, \ldots, r$. Then according to [10], 2.4, there is a unique reduced homotopy class $\kappa'_M(f)$ as in diagram 3.2 above such that $\text{quot}^* \circ \text{incl}_*(\kappa'_M(f))$ equals the base point preserving version $\kappa^b_M(f)$ of the κ-invariant of f (compare 1.2). For $s \geq 0$ define

$$\mu^{(s)}_M(f) \ := \ H_s(\kappa'_M(f)) \in \bigoplus^{u_{r,s}} \pi^S_{|p|-s(n-1)-(r-1)(m-2)-1} \tag{3.4}$$

(cf. 2.7).

Note that $\mu^{(0)}_M(f)$ coincides with the invariant $\mu_M(f)$ discussed in [10], § 2. Indeed, the homomorphism c_a used there is induced by the map which collapses S^n; hence $H_0 = h \circ c_a$ (if $n = m-1$ we assume here that the arcs involved in c_a converge to $S^{m-1} \times \{+\infty\}$).

The main results 2.12 and 2.27 of the previous section, together with lemma 2.28, now imply

Theorem 3.5. *If* $p_1 + \cdots + p_r \leq r(m-2)$*, then the base point preserving* κ*-invariant* $\kappa^b_M(f)$ *of a* κ_M*-Brunnian link map* f *into* $M = S^n \times \mathbb{R}^{m-n}$, $1 \leq n < m \geq 3$*, contains precisely as much information as the sequence* $\{\mu^{(s)}_M(f)\}_{s \geq 0}$ *which starts with* $\mu_M(f)$*.*

What is the geometric meaning of the remaining “higher μ-invariants” in this sequence?

In order to answer this question, fix any orientation preserving smooth embedding

$$\eta \ : \ S^n \times B^{m-n} \ \hookrightarrow \ \mathbb{R}^m \tag{3.6}$$

where B^{m-n} is the compact $(m-n)$–dimensional unit ball with interior $\overset{\circ}{B}{}^{m-n} \cong \mathbb{R}^{m-n}$. Thus η defines an inclusion of $S^n \times \mathbb{R}^{m-n}$ into $\mathbb{R}^m$, as well as a meridian

$$\eta_z \ : \ S^{m-n-1} \ = \ \{z\} \times \partial B^{m-n} \ \overset{\eta|}{\hookrightarrow} \ \mathbb{R}^m$$

for every $z \in S^n$.

Now, given any link map in $S^n \times \mathbb{R}^{m-n}$, add s meridians $\eta_{z_1}, \ldots, \eta_{z_s}$ (at pairwise distinct points $z_1, \ldots, z_s \in S^n$). Up to link homotopy the resulting augmented link map

$$f^{(s)} \ = \ \left(\coprod_{j=1}^{s} \eta_{z_j} \right) \amalg \eta \circ f \ : \ \coprod_{j=1}^{s} S^{m-n-1} \amalg \coprod_{i=1}^{r} S^{p_i} \ \longrightarrow \ \mathbb{R}^m \quad , \tag{3.7}$$

$s \geq 0$, depends only on the link homotopy class of f in M.

Theorem 3.8. *Assume* $|p| \leq r(m-2)$ *and* $s \geq 0$*. Let* $f : \amalg S^{p_i} \longrightarrow M = S^n \times \mathbb{R}^{m-n}$ *be a* κ_M*–Brunnian and base point preserving link map.*

If the augmented link map $f^{(s)}$ *in* $\mathbb{R}^m$ *(cf. 3.7) is* $\kappa_{\mathbb{R}^m}$*–Brunnian (or, equivalently,* $\mu^{(t)}_M(f) = 0$ *for all* $t < s$ *), then for all* $\gamma \in \Sigma_{s+r-2}$

$$\mu^{(s)}_{M,\gamma}(f) \ := \ H_{s,\gamma}(\kappa'_M(f)) = \mu_{\mathbb{R}^m,\gamma}(f^{(s)})$$

(at least up to a fixed sign), and hence, in particular, $\mu_M^{(s)}(f)$ *depends only on the base point free link homotopy class of* f *in* M*.*

Proof. Since f is κ_M–Brunnian, the product map $\widehat{f}$ (cf. 1.2) can be deformed in a base point preserving fashion until it factors through a map of the form

$$F_{s+r}^{(s)} \;:\; S^{|p|} \;\longrightarrow\; S^n \vee \bigvee^{r-1} S^{m-1} \;\subset\; M - \{y_1^0, \dots, y_{r-1}^0\} \; (\underset{\eta}{\hookrightarrow} \mathbb{R}^m).$$

This extends to the new link map

$$F^{(s)} \;=\; \amalg\, F_i^{(s)} \;:\; \coprod_{j=1}^{s} S^{m-n-1} \;\amalg \coprod_{i=1}^{r-1} S^0 \;\amalg\; S^{|p|} \;\longrightarrow\; \mathbb{R}^m \tag{3.9}$$

which involves also the meridians $F_j^{(s)} = \eta_{z_j}$, $j = 1, \dots, s$ (compare 3.7), and $F_{s+i}^{(s)}$ which map the base point $*$ of S^0 to a suitable value in $\eta(S^n \times \partial D^{m-n})$ and the other point of S^0 to y_i^0, $i = 1, \dots, r-1$. (Recall that 0-dimensional link components present no problems in $\mathbb{R}^m$, cf. [9]). If $F^{(s)}$ is $\kappa_{\mathbb{R}^m}$-Brunnian then so is $f^{(s)}$, and the $\kappa'_{\mathbb{R}^m}$-invariants of $F^{(s)}$ and $f^{(s)}$ coincide (they vanish if and only if $F^{(s+1)}$ or, equivalently, $f^{(s+1)}$ is $\kappa_{\mathbb{R}^m}$-Brunnian); on the other hand, since $F^{(s)}$ is link homotopic to $e_*((\mathrm{pinch}_s \vee \mathrm{id}) \circ F_{s+r}^{(s)})$, it follows from theorem 6.1 in [9] that for all $\gamma \in \sum_{s+r-2}$

$$h_\gamma(\kappa'_{\mathbb{R}^m}(F^{(s)})) \;=\; \pm h_\gamma((\mathrm{pinch}_s \vee \mathrm{id}) \circ F_{s+r}^{(s)}) \;.$$

But these two expressions are $\mu_{\mathbb{R}^m,\gamma}(f^{(s)}$ and $H_{s,\gamma}(\kappa'_M(f))$, resp. We conclude, in particular, that $\mu^{(s)}(f)$ vanishes if and only if $\kappa'_{\mathbb{R}^m}(f^{(s)})$ does (use (24) in [9]). An induction over s now completes the proof. ■

Recall that the base point free homotopy class $\kappa_M(f)$ (cf. 1.2) is trivial precisely if its base point preserving analogue $\kappa_M^b(f)$ is.

Corollary 3.10. *Assume* $p_1, \dots, p_r \le m-2$.

Then for all link maps $f : \amalg S^{p_i} \longrightarrow M = S^n \times \mathbb{R}^{m-n}$ *the following two conditions are equivalent:*

(i) $\kappa_M(f)$ *is trivial* (*cf.* 1.2) *; and*
(ii) *the component maps* $f_i : S^{p_i} \longrightarrow M \sim S^n$ *are nulhomotopic,* $i = 1, \dots, r$, and *the invariant* $\kappa_{\mathbb{R}^m}(f^{(s)})$ *of the augmented link map* $f^{(s)}$ *is trivial for every* $s \ge 0$.

Proof. Recall from [9], theorem 4.2, that the second part of condition (ii) holds if and only if all (the consecutively defined) $\mu_{\mathbb{R}^m}$–invariants of all sub-link maps of $f^{(s)}$, $s \ge 0$, vanish.

Clearly, (i) implies (ii). Assume that we have proved the converse inductively for link maps with up to $r-1$ components. Then (ii) implies that f is κ_M–Brunnian and, by theorem 3.8, even that $\kappa_M(f)$ is trivial (use induction over s and theorem 3.5) ■

Remark 3.11. If $n \ge 2$, then M is 1–connected; thus $\kappa_M^b(f)$ (and hence all $\mu_M^{(s)}(f)$, when defined) are invariant even under *base point free* link homotopies of

f (compare [10], § 3). Moreover, the higher μ-invariants $\mu_M^{(s)}(f)$, $s \geq 1$, are in general truly new and not already determined by the invariant $\mu_M^{(0)}(f) = \mu_M(f) = \widetilde{\mu}_M(f)$ studied in [10]. This is illustrated e.g. by example 5.9 of [10] where $f^{(1)}$ is a higher dimensional version of the classical Borromean link and $\mu_M(f) = 0$ but $\mu^{(1)}(f) \neq 0$.

If $n = 1$ and $p_1, \ldots, p_r \leq m-2$, the base point preserving κ–invariant $\kappa_M^b(f)$ (cf. 3.2) of a κ_M-Brunnian link map f is precisely as strong as

$$\widetilde{\mu}_M(f) \ := \ \{h_{(g),\gamma}(\kappa'_M(f))\} \in \bigoplus_{((g),\gamma)\in\mathbb{Z}^{r-1}\times\Sigma_{r-2}} \pi^S_{|p|-(r-1)(m-2)-1}$$

(see 2.24 as well as [10], 2.11 and 5.6) on one hand, and as the sequence

$$\left\{\mu_M^{(s)}(f) \ \in \ \bigoplus_{\gamma\in\Sigma_{r,s}} \pi^S_{|p|-(r-1)(m-2)-1}\right\}_{s\geq 0}$$

on the other hand (cf. 3.5). The transition to the base point free homotopy invariant $\kappa_M(f)$ amounts to dividing out the translating action of $\mathbb{Z}^{r-1}$ on the indices $((g),\gamma)$ of the factors in $\widetilde{\mu}_M(f)$ (see [10], 3.3 and 5.6). On the other hand, $\mu_M^{(s)}(f)(= \mu_{\mathbb{R}^m}(f^{(s)})$ is invariant under base point free link homotopies as soon as all $\mu_{\mathbb{R}^m}(f^{(t)})$, $t < s$, are successively defined and trivial (or, equivalently, the "preceding" higher μ-invariants $\mu_M^{(0)}(f), \ldots, \mu_M^{(s-1)}(f)$ vanish; compare 3.8).

4. Connected sums and the extended definition of the invariants

It is often possible and useful to define κ- and μ-invariants for arbitrary (not necessarily κ-Brunnian) link maps.

Assume that $1 \leq p_1, \ldots, p_r \leq m-3$. Consider a smooth connected target manifold of the form $M^m = M' \times \mathbb{R}$ where the base points $y_1^0, \ldots, y_r^0$ lie in $M' \times \{0\}$. Then there is a well-defined "connected sum" addition on the set $BLM_{(p)}(M)$ of base point preserving link homotopy classes of link maps

$$f \ = \ \amalg f_j \ : \ \amalg S^{p_j} \ \longrightarrow \ M \qquad : \tag{4.1}$$

just deform the summands f^+ and f^- into $M' \times [0,\infty)$ and $M' \times (-\infty, 0]$, resp., and form the obvious sum $f^+ + f^-$. If even $M = M'' \times \mathbb{R}^2$ this operation makes $BLM_{(p)}(M)$ into an associative commutative semigroup with unit (provided $\pi_1(M)$ is abelian whenever $\min\{p_i\} = 1$). In this case successive addition of sublink maps of the form

$$f_{i_1} \ \amalg \cdots \amalg \ f_{i_{s-1}} \ \amalg \ f_{i_s} \circ \text{ reflection} \quad , \quad s < r,$$

(together with $r-s$ constant component maps) allows us to "retract" (in a κ-theoretical sense) arbitrary link maps to κ-Brunnian ones and then to apply our invariants. For details compare [9], pp. 311–312.

We obtain in particular

Proposition 4.2. *Assume* $1 \le p_1, \ldots, p_r \le m-3$ *and* $1 \le n < m$. *Then there are additive invariants*

$$\kappa'_M(f) \;\in\; \pi_{|p|}\left(S^n \vee \bigvee_{j=1}^{r-1} S^{m-1}\right)$$

and $\mu^s_M(f) := H_s(\kappa'_M(f)), s = 0, 1, \ldots,$ *canonically defined for* every *base point preserving link map* $f \;:\; \amalg S^{p_j} \longrightarrow M = S^n \times \mathbb{R}^{m-n}$. *We have:*

(i) *if* f *is* κ_M*-Brunnian, these extended definitions agree with the previous ones (cf.* 3.2 *and* 3.4*);*

(ii) *these invariants vanish whenever at least one of the components maps* f_j *of* f *is constant.*

The case $n = m-1$ is taken care of by the isomorphisms $BLM_{(p)}(S^{m-1} \times \mathbb{R}) \cong BLM_{(p)}((S^{m-1} - \{*\}) \times \mathbb{R}) \cong BLM_{(p)}(\mathbb{R}^m)$.

Recall also that in the simply-connected case $n \le 2$ base point preserving and base point free link homotopy theory coincide here.

Next we extend theorem 3.8.

Theorem 4.3. *Under the assumption of proposition* 4.2 *the identity*

$$\mu^{(s)}_{M,\gamma}(f) \;:=\; H_{s,\gamma}(\kappa'_M(f)) \;=\; \mu_{\mathbb{R}^m,\gamma}(f^{(s)})$$

holds (*at least up to a fixed sign*) *for* every *link map* $f : \amalg S^{p_j} \longrightarrow M = S^n \times \mathbb{R}^{m-n}$ (*not necessarily* κ*-Brunnian; but base point preserving if* $n = 1$) *and for all* $s \ge 0$ *and* $\gamma \in \sum_{s+r-2}$.

Proof. Adding suitable link maps in M with at least one constant component changes neither $\mu^{(s)}_M(f)$ nor $\mu_{\mathbb{R}^m}(f^{(s)}$. Thus we may assume that f is κ_M-Brunnian. In order to make also $f^{(s)}$ $\kappa_{\mathbb{R}^m}$-Brunnian we habe to add successively further link maps in $\mathbb{R}^m$ of the form

$$\coprod_{j=1}^{s} \eta_{z_j} \;\amalg\; \eta \circ f_1 \amalg \cdots \amalg \eta \circ f_{r-1} \;\amalg\; \eta \circ f_r \circ (\text{reflection})^i \;, \quad i = 0 \text{ or } 1 \;,$$

(cf. 3.7), but with at least one component η_{z_j} replaced by a constant map or, equivalently, the remaining components deformed in such a way that they do not intersect the slice $\eta(\{z_j\} \times B^{m-n}) \subset \mathbb{R}^m$ (cf. 3.6). This changes neither $\mu^{(s)}_M(f) = h \circ (\text{pinch}_s \vee \text{id})(F^{(s)}_{s+r})$ (cf. 3.9) nor $\mu_{\mathbb{R}^m}(f^{(s)})$, and as in the proof of 3.8 above these two terms agree by the projectability theorem 5.2 in [9]. ∎

5. Linking coefficients and stable suspensions

Theorem 5.1. *Assume that* $1 \le p_1, \ldots, p_r \le m-3$ *and* $1 \le n \le m-1$. *Let* $f = \amalg^r f_j \;:\; \amalg^r S^{p_j} \longrightarrow M = S^n \times \mathbb{R}^{m-n}$ *be a link map (base point preserving if*

$n = 1$) such that $f_1 \amalg \cdots \amalg f_{r-1}$ is an embedding (and hence f_r determines the linking coefficient

$$\lambda(f) \ \in \ \pi_{p_r}\left(S^n \vee \bigvee_{j=1}^{r-1} S^{m-p_j-1} \right).$$

Then the following (systems of) invariants contain an equal amount of information:

(i) $\kappa'_M(f)$;
(ii) $\{\mu_M^{(s)}(f) = H_s(\kappa'_M(f)\}_{s\geq 0}$;
(iii) $\{H_s(\lambda(f))\}_{s\geq 0}$;
(iv) *the stable suspensions of all the Hilton components of $\lambda(f)$ corresponding to those basic Whitehead products which involve each meridian sphere S^{m-p_j-1} precisely once, $j = 1, \dots, r-1$.*

Proof. It is a well-known consequence of the Thom isomorphism and Whitehead theorems that the inclusion incl of the wedge

$$W \ := \ S^n \ \vee \ \bigvee_{j=1}^{r-1} \ S_j^{m-p_j-1} \tag{5.2}$$

(formed essentially by a core $S^n \times \{*\} \subset S^n \times \mathbb{R}^{m-n}$ and by meridians to $\underline{f} := f_1 \amalg \cdots \amalg f_{r-1}$) into the link complement of $\underline{f}$ induces isomorphisms of homotopy groups up to dimension $m-3$ (compare also [10], 4.6). The Hilton components mentioned above in (iv) constitute the $\pi''_*(W)$-part $\lambda''(f)$ of $\lambda(f) := \mathrm{incl}_*^{-1}([f_r])$ (cf. 2.9, 2.15, and diagramm 5.3 below).

After a suitable isotopy the images of $f_1, \dots, f_{r-1}$ intersect $S^n \times \mathbb{R}^{m-n-1} \times [0, \infty)$ in small disjoint half-spheres and we may assume that W lies in $S^n \times \mathbb{R}^{m-n-1} \times [0, \infty)$. Thus the link homotopy class $[f]$ is the sum (cf. § 4) of $[f_1 \amalg \dots, \amalg f_{r-1} \amalg \mathrm{constant}]$ with the class $e_*(\lambda(f))$ consisting of $r-1$ standard spheres in parallel hyperplanes in a suitable coordinate neighbourhood and of $\lambda(f)$ (cf. section 4 of [10]). Moreover the "retraction" procedure discussed in § 4 has just the effect of replacing $\lambda(f)$ by its $\pi'_*(W)$-part $\lambda'(f)$ which involves each meridian at least once (compare 2.3). By proposition 4.2 we conclude

$$\kappa'_M(f) \ = \ \kappa'_M \circ e_*(\lambda(f)) \ = \ \kappa'_M \circ e_*(\lambda'(f)) \ .$$

For $n \geq 2$ the proof of our theorem follows from diagram 5.3. Here $\Sigma w_{\ell *}, \Sigma \overline{w}_{\ell *}$

and H denote (corresponding) Hilton isomorphisms and Hopf homomorphisms; moreover,

$$k_s \ := \ s(n-1)+(r-1)(m-2)-|p|+p_r+1 \ .$$

Theorem 4.10 in [10] and results 2.11 and 2.12 above imply that actually

$$\kappa'_M(f) \ = \ \kappa'_M \circ e_*(\lambda''(f))$$

and that each part of diagram 5.3 commutes up to an automorphism of the target group (which is compatible with the s-grading). In particular, for each $s \geq 0$ the invariants

$$\mu_M^{(s)}(f) \qquad\qquad \text{and} \qquad\qquad H_s(\lambda(f)) \ = \ H_s(\lambda''(f))$$

are related by an invertible matrix with integer coefficients.

A similar diagram, based on theorem 2.27, yields a proof of our claim also in the case $n = 1$. ■

In view of the last theorem nontriviality, surjectivity, or injectivity results for the stable suspensions E^∞ have corresponding implications for our invariants. In particular, in the stable dimension range we obtain for the setting of 5.1

Corollary 5.4. *If* $p_r/2 \leq \sum_{j=1}^{r-1}(m-p_j-2)$*, then* $\lambda''(f)$ (*i. e. the* $\pi''(W)$*-part of the linking coefficient* $\lambda(f)$, *cf.* 2.9 *and* 2.15) *is invariant under link homotopies* (*assumed to preserve base points in case* $n=1$) *and contains precisely as much information as* $\kappa'_M(f)$.

If $n \geq 2$*, then for every natural number* s *satisfying*

$$\frac{p_r}{2} \ \leq \ s(n-1) \ + \ \sum_{j=1}^{r-1}(m-p_j-2)$$

the $\pi''(W;s)$*-part* (*cf.* 2.9) *of* $\lambda(f)$ *is a* (*base point free*) *link homotopy invariant of* f *and precisely as strong as* $\mu_M^{(s)}(f)$*; moreover* $\mu_M^{(s)}$ *assumes all values in its target group* (*cf.* 3.4).

If f is *homotopy Brunnian* (i.e. every proper sub-link map is nulhomotopic), then under certain dimension conditions the only relevant contribution comes from $\lambda''(f)$ (cf. 2.28 and compare [10], 4.6).

Corollary 5.5. *Assume* $p_1, \dots, p_r \le m-3$ *and*

$$\begin{aligned} |p| \ &\le\ (r-1)(m-2) + \frac{p_r}{2} && \textit{and} \\ |p| \ &\le\ r(m-2) - p_j && , \quad j = 1, \dots, r-1. \end{aligned}$$

Then the (base point preserving) link homotopy class of a homotopy Brunnian link map $f : \coprod_{j=1}^r S^{p_j} \longrightarrow M = S^n \times \mathbb{R}^{m-n}$ *which embeds* $\coprod_{j=1}^{r-1} S^{p_j}$ *is completely determined by* $\kappa'_M(f)$ *or, equivalently, by* $\{\mu_M^{(s)}(f)\}_{s\ge 0}$.

If $n \ge 2$*, this establishes an isomorphism from the semigroup of such classes onto* $\bigoplus_{s\ge o} (\pi^S_{|p|-s(n-1)-(r-1)(m-2)-1})^{u_{r,s}}$.

Note that the dimension conditions above are satisfied e.g. if $p_1 = \dots = p_r = m-3 \le r$.

Finally we prove our claims concerning the examples in the introduction. To begin with, assume $p_1 = \dots = p_r = m-3 = 3$.

Fix at first $n \ge 2$ and put $M = M_n$. Then

$$\mu(f) = \Big(\{[f_\ell]\}, \{(\mu_M^{(0)} \oplus \mu_M^{(1)})(f_k \amalg f_\ell)\}, \{\mu_M^{(0)}(f_j \amalg f_k \amalg f_\ell)\}\Big)$$

lies in $(\pi_3(S^n))^r \oplus (\mathbb{Z}_2 \oplus \pi^S_{2-n})^{\binom{r}{2}} \oplus (\pi^S_0)^{\binom{r}{3}}$; here the first $\mu_M^{(0)}$ measures just the α-invariants (= generalized linking numbers in $\pi^S_1 = \mathbb{Z}_2$) of all 2-component sub-link maps of f, when included into $\mathbb{R}^6$ (cf. 3.8 and [10], 2.14).

After a suitable link homotopy we may assume that f is an embedding (add a small copy of f_j in a small 6-ball, $j = 1, \dots, r$, and apply the Whitney trick). As in the proof of theorem 5.1 we obtain therefore

$$[f] = [f_1 \amalg \cdots \amalg f_{r-1} \amalg *] + e_*(\lambda_1(f)) + e_*(\lambda_2(f)) + e_*(\lambda'_2(f)) + e_*(\lambda_3(f))$$

where f_r determines the linking coefficient $\lambda(f)$ in

$$\pi_3(S^n \vee \bigvee^{r-1} S^2) \cong \pi_3(S^n) \oplus \bigoplus_{k=1}^{r-1} (\pi_3(S^2) \oplus \pi_3(S^{n+1})) \oplus \bigoplus_{1\le j<k<r} \pi_3(S^3)$$

and $\lambda_1(f), \dots$ denote its Hilton components corresponding to the basic Whitehead products $\iota_0, \{\iota_k\}, \{[\iota_0, \iota_k]\}$ and $\{[\iota_j, \iota_k]\}$, resp. Their values under e_* are detected by μ (cf. proposition 2.11 and [10], theorem 4.10) and yield the contribution for $\ell = r$. Note here that, given $u \in \pi_3(S^2) \cong \mathbb{Z}$, $e_*(\iota_k(u))$ lies in an open ball $B \subset S^n \times \mathbb{R}^{6-n} \subset \mathbb{R}^6$ and recall the isomorphism

$$\alpha \ : \ LM_{3,3}(\mathbb{R}^6) \ \xrightarrow{\cong} \ \pi^S_1 \ = \ \mathbb{Z}_2$$

(cf. [8], proposition F, or [4], theorem 1) which distinguishes $e_*(\iota_k(u))$ according to the stable suspension (or, equivalently, parity) of u.

Induction over r completes the proof for $n \geq 2$.

If $n = 1$, approximate f by a self-transverse immersion and add the nulhomotopic link maps $* \amalg \cdots \amalg$ reflection $\circ f_j \amalg * \amalg \cdots \amalg *$, $j = 1, \dots r$; then we can cancel corresponding double points without complications arising from $\pi_1(M_1)$. The linking coefficient $\lambda(f)$ of the resulting embedding lies now in the group

$$\pi_3(S^1 \vee \bigvee^{r-1} S^2) \;\cong\; \bigoplus_{k=1}^{r-1} \left(\bigoplus_{g \in \mathbb{Z}} \pi_3(S^2) \right) \oplus \bigoplus_{1 \leq j < k < r} \left(\bigoplus_{(g,g') \in \mathbb{Z}^2} \pi_3(S^3) \right) \oplus \cdots$$

with basic Whitehead products $\iota_{k,g}$ (cf. 2.13 – 2.14), $[\iota_{j,g}, \iota_{k,g'}]$ and $[\iota_{j,g}, \iota_{j,h}]$, $g < h \in \mathbb{Z}$. The products of the last type can be neglected since $e_*([\iota_{j,g}, \iota_{j,h}])$ is trivial by proposition 4.13 in [10]. Also the contributions coming from $e_* \circ \iota_{j,g}$ factor through the stable suspension homomorphism from $\pi_3(S^2)$ to $\mathbb{Z}_2$ again and then are detected by μ (cf. theorem 2.27 and [10], theorem 4.10). So are the remaining Hilton parts of $\lambda(f)$ which yield 3-component sub-link maps of f. By induction over r we see that each pair $k < \ell$ (and each triple $j < k < \ell$, resp.) of indices between 1 and r contributes the direct summand

$$\bigoplus_{g \in \mathbb{Z}} \mathbb{Z}_2 \quad \left(\text{and} \quad \bigoplus_{(g,g') \in \mathbb{Z}^2} \mathbb{Z} \quad , \textit{resp.} \right)$$

to $BLM_{3,\dots,3}(M_1)$. The inclusion $M_1 \subset M_2$ induces the identification of the meridians $\iota_{j,g}$, $g \in \mathbb{Z}$, with the single meridian ι_j, $j = 1, \dots, r$, and hence the summation homomorphisms on $\mathbb{Z}_2^\infty$ and $\mathbb{Z}^\infty$. ■

The claim in the second example in the introduction follows from theorem 5.1 above and from proposition 4.13 in [10].

Acknowledgement. It is a pleasure to thank Uwe Kaiser for stimulating discussions.

References

[1] A. Bartels and P. Teichner, *All two dimensional links are null homotopic*, Geometry & Topology **3** (1999), 235–252.
[2] U. Dahlmeier, *Verkettungshomotopien in Mannigfaltigkeiten* (1993), Doctoral dissertation, Siegen.
[3] N. Habegger, *Knots and links in codimension greater than* 2, Topology **25** (1986), no. 3, 253–260.
[4] N. Habegger and U. Kaiser, *Link homotopy in the* 2*–metastable range*, Topology **37** (1998), no. 1, 75–94.
[5] D. Hacon, *Embeddings of* S^p *in* $S^1 \times S^q$ *in the metastable range*, Topology **7** (1968), 1–10.
[6] P. Hilton, *On the homotopy groups of the union of spheres*, J. of the London Math. Soc. **30** (1955), 154–172.
[7] U. Kaiser, *Link theory in manifolds*, Lect. Notes in Math. **1669** (1997), Springer.
[8] U. Koschorke, *On link maps and their homotopy classification*, Math. Ann. **286** (1990), 753–782.
[9] ———, *A generalization of Milnor's* μ*–invariants to higher-dimensional link maps*, Topology **36** (1997), no. 2, 301–324.

[10] ———, *Link maps in arbitrary manifolds and their homotopy invariants*, J. of Knot Theory and its Ramifications (to appear).

[11] J. Milnor, *Link groups*, Ann. of Math. **59** (1954), 177–195.

Universität Siegen, Emmy Noether Campus, Walter-Flex-Str. 3, D-57068 Siegen, Germany

E-mail address: koschorke@mathematik.uni-siegen.de